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A GENERAL FORMULATION OF LINEAR SPLITTING METHODS FOR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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A general formulation of linear splitting methods for ordinary and partial differential equations

by

P.J. van der Houwen & J.G. Verwer

ABSTRACT

The main purpose of the paper is to present a unified treatment of one-step integration methods for systems of ordinary differential equations $\frac{d\vec{y}}{dx} = f(\vec{y})$, $\vec{f}$ satisfying a linear splitting $\vec{f}(\vec{y}) = \sum_{i=1}^{k} \vec{f}_i(\vec{y})$. The emphasis is on systems which originate from semi-discretization of time dependent partial differential equations. A class of integration methods is defined, which contains all known splitting methods, such as alternating direction and locally one-dimensional methods, provided these methods are formulated for semi-discretized equations satisfying a linear splitting.

KEY WORDS & PHRASES: *Numerical Analysis, Ordinary differential equations, Partial differential equations, Method of lines, Splitting methods.*

CONTENTS

1. INTRODUCTION

In the numerical treatment of partial differential equations the idea of splitting is to break down a complicated multi-dimensional process into a series of one-dimensional, and less complicated processes. Well-known splitting methods (also referred to as fractional step methods) are the alternating direction methods, (PEACEMAN & RACHFORD [12], DOUGLAS [2]), the locally one-dimensional methods (YANENKO [16, p.23]), and the hop-scotch methods (GOURLAY [6,8]). Since the first papers of Peaceman, Rachford and Douglas, many authors has written contributions in this field, mainly on initial boundary value problems. In these contributions, splitting methods are generally formulated and analysed as so-called direct grid methods. The idea of splitting can also be applied in conjunction with *the method of lines*, an approach which is preferred by the authors. In order to elucidate this point of view we mention three points:

1°. The character of a splitting difference scheme, when formulated as a direct grid method, is governed almost wholly by the type of the time discretization. The type of the space discretization is of lesser importance (cf. the discussion in GOURLAY & MITCHELL [7]).

2°. In the analysis and application of direct grid splitting methods the boundary treatment takes an important place and often causes severe problems (cf. the discussion in GOURLAY [8]). Point 1° suggests that in the analysis and application of a splitting difference scheme for initial boundary value problems the boundary treatment is better handled apart. This is achieved by applying the method of lines.

3°. By defining splitting methods for semi-discretized partial differential equations, or in general for systems of ordinary differential equations, it is possible to give a unified treatment of these methods. When defining these methods as direct grid methods, this is hardly possible.

To our opinion point 3° is very important. Consequently, the main purpose of the present paper is to formulate a wide class of splitting methods for ordinary differential equations, which contains all known one-step splitting methods for a general class of problems. We will consider

non-linear systems of ordinary differential equations of the autonomous form

$$(1.1) \qquad \frac{d\vec{y}}{dx} = \vec{f}(\vec{y}),$$

of which the right-hand side $\vec{f}(\vec{y})$ is supposed to be linearly splitted into k terms, i.e.

$$(1.2.) \qquad \vec{f}(\vec{y}) = \sum_{i=1}^{k} \vec{f}_i(\vec{y}),$$

where the vector functions $\vec{f}_i$ are supposed to be of sufficient differentiability. The restriction to autonomous form is made for notational convenience and is not essential, i.e. results can be easily extended to equations of the non-autonomous form

$$(1.3) \qquad \frac{d\vec{y}}{dx} = \vec{f}(x,\vec{y}).$$

In the definition of our integration formulas no further a priori knowledge of the functions $\vec{f}_i$ is assumed. In the paper, several examples of schemes are given, which may be recognized as existing schemes, or as generalizations, when we confine ourselves to a particular problem class.

For the preparation of this paper the book of YANENKO [16] on fractional step methods and the survey of GOURLAY [8] were very useful.

2. THE CLASS OF SPLITTING METHODS

Let $\vec{y}_n$ denote the numerical approximation at $x = x_n$. Let h_n denote the n-th integration stepsize, i.e. $h_n = x_{n+1} - x_n$. The class of splitting methods discussed in this paper is then defined by the m-th stage, one-step scheme

$$\vec{y}_{n+1}^{(0)} = \vec{y}_n,$$

$$(2.1)\qquad \vec{y}_{n+1}^{(j)} = \vec{y}_n + h_n \sum_{\ell=0}^{j} \sum_{i=1}^{k} \lambda_{j\ell i}\, \vec{f}_i(\vec{y}_{n+1}^{(\ell)}), \qquad j = 1(1)m,$$

$$\vec{y}_{n+1} = \vec{y}_{n+1}^{(m)}.$$

The parameters $\lambda_{j\ell i}$ serve to make this scheme a consistent approximation to the differential equation (1.1). But, in particular, they should be used to exploit the splitting property (1.2) in order to obtain a computationally efficient and numerically stable process. The calculation of the approximation $\vec{y}_{n+1}^{(j)}$ is straightforward when the parameters λ_{jji}, $i = 1(1)k$, are equal to zero. Otherwise, an algebraic system of generally non-linear equations should be solved. Observe that for $k = 1$, i.e. when no splitting is performed, scheme (2.1) reduces to a m-th stage, semi-explicit Runge-Kutta scheme (see BUTCHER [1] or NORSETT [11]). In the sequel it is always assumed that $k \geq 2$.

2.1. The order conditions

In this section we will derive the order conditions for scheme (2.1) up to order $p = 3$. To that end we formally expand $\vec{y}_{n+1}$ in a Taylor series at the point x_n:

$$(2.2)\qquad \vec{y}_{n+1} = \vec{y}_n + h_n \sum_{\ell=0}^{m} \sum_{i=1}^{k} \lambda_{m\ell i} \Big[\vec{f}_i(\vec{y}) + (\vec{y}_{n+1}^{(\ell)} - \vec{y}_n \cdot \nabla)\vec{f}_i(\vec{y}) + \tfrac{1}{2}(\vec{y}_{n+1}^{(\ell)} - \vec{y}_n \cdot \nabla)^2 \vec{f}_i(\vec{y})\Big]_{\vec{y}_n} + O(h_n^4).$$

Here, $(\ .\ )$ denotes the usual innerproduct and ∇ is the gradient operator $(\partial/\partial y^j)$, where y^j is the j-th component of $\vec{y}$. In a similar way, for $\ell = 1(1)m$,

$$\vec{y}_{n+1}^{(\ell)} - \vec{y}_n = h_n \sum_{r=0}^{\ell} \sum_{j=1}^{k} \lambda_{\ell r j} \Big[\vec{f}_j(\vec{y}) + (\vec{y}_{n+1}^{(r)} - \vec{y}_n \cdot \nabla)\vec{f}_j(\vec{y})\Big]_{\vec{y}_n} + O(h_n^3).$$

Substitution into (2.2) yields

$$(2.2') \qquad \vec{y}_{n+1} = \vec{y}_n + h_n \sum_{\ell=0}^{m} \sum_{i=1}^{k} \lambda_{m\ell i} \Big\{ \vec{f}_i(\vec{y}) + h_n \sum_{r=0}^{\ell} \sum_{j=1}^{k} \lambda_{\ell r j} ([\vec{f}_j(\vec{y}) + (\vec{y}_{n+1}^{(r)} - \vec{y}_n \cdot \nabla)\vec{f}_j(\vec{y})]_{\vec{y}_n} \cdot \nabla)\vec{f}_i(\vec{y}) + \tfrac{1}{2}h_n^2 \Big(\sum_{r=0}^{\ell} \sum_{j=1}^{k} \lambda_{\ell r j} \vec{f}_j(\vec{y}_n) \cdot \nabla \Big)^2 \vec{f}_i(\vec{y}) \Big\}_{\vec{y}_n} + O(h_n^4),$$

where $\lambda_{0rj} = 0$ for all r and j.
Finally, by substituting

$$\vec{y}_{n+1}^{(r)} - \vec{y}_n = h_n \sum_{s=0}^{r} \sum_{t=1}^{k} \lambda_{rst} \vec{f}_t(\vec{y}_n) + O(h_n^2)$$

and rearranging terms in (2.2') we arrive at the expansion

$$(2.2'') \qquad \vec{y}_{n+1} = \vec{y}_n + h_n \sum_{\ell=0}^{m} \sum_{i=1}^{k} \lambda_{m\ell i} \vec{f}_i(\vec{y}_n) + h_n^2 \sum_{\ell=0}^{m} \sum_{i=1}^{k} \sum_{r=0}^{\ell} \sum_{j=1}^{k} \lambda_{m\ell i}\lambda_{\ell r j} (\vec{f}_j(\vec{y}_n) \cdot \nabla)\vec{f}_i(\vec{y})|_{\vec{y}_n} + h_n^3 \sum_{\ell=0}^{m} \sum_{i=1}^{k} \lambda_{m\ell i} \Big\{ \tfrac{1}{2}\Big(\sum_{r=0}^{\ell} \sum_{j=1}^{k} \lambda_{\ell r j} \vec{f}_j(\vec{y}_n) \cdot \nabla \Big)^2 \vec{f}_i(\vec{y}) + \sum_{r=0}^{\ell} \sum_{j=1}^{k} \sum_{s=0}^{r} \sum_{t=1}^{k} \lambda_{\ell r j}\lambda_{rst} ((\vec{f}_t(\vec{y}_n) \cdot \nabla)\vec{f}_j(\vec{y})|_{\vec{y}_n} \cdot \nabla)\vec{f}_i(\vec{y}) \Big\}_{\vec{y}_n} + O(h_n^4).$$

The formal expansion, up to order three, of the local analytical solution $\vec{y}(x)$ through the point $(\vec{x}_n, \vec{y}_n)$ is given by

$$(2.3)\qquad \vec{y}(x_n+h_n) = \vec{y}_n + h_n \sum_{i=1}^{k} \vec{f}_i(\vec{y}_n) +$$

$$+ \frac{1}{2} h_n^2 \sum_{i=1}^{k} \sum_{j=1}^{k} (\vec{f}_j(\vec{y}_n) \cdot \nabla)\vec{f}_i(\vec{y})\Big|_{\vec{y}_n} +$$

$$+ \frac{1}{6} h_n^3 \sum_{i=1}^{k} (\sum_{j=1}^{k} \vec{f}_j(\vec{y}) \cdot \nabla)^2 \vec{f}_i(\vec{y})\Big|_{\vec{y}_n} +$$

$$+ O(h_n^4).$$

Using the relation

$$(\vec{f}(\vec{y}) \cdot \nabla)^2 \vec{f}_i(\vec{y})\Big|_{\vec{y}_n} = (\vec{f}(\vec{y}_n) \cdot \nabla)^2 \vec{f}_i(\vec{y})\Big|_{\vec{y}_n} + ((\vec{f}(\vec{y}_n) \cdot \nabla)\vec{f}(\vec{y})\Big|_{\vec{y}_n} \cdot \nabla)\vec{f}_i(\vec{y})\Big|_{\vec{y}_n}$$

expansion (2.3) may be written as

$$(2.3')\qquad \vec{y}(x_n+h_n) = \vec{y}_n + h_n \sum_{i=1}^{k} \vec{f}_i(\vec{y}_n) +$$

$$+ \frac{1}{2} h_n^2 \sum_{i=1}^{k} \sum_{j=1}^{k} (\vec{f}_j(\vec{y}_n) \cdot \nabla)\vec{f}_i(\vec{y})\Big|_{\vec{y}_n} +$$

$$+ \frac{1}{6} h_n^3 \sum_{i=1}^{k} \Big\{ (\sum_{j=1}^{k} \vec{f}_j(\vec{y}_n) \cdot \nabla)^2 \vec{f}_i(\vec{y}) +$$

$$+ \sum_{j=1}^{k} \sum_{t=1}^{k} ((\vec{f}_t(\vec{y}_n) \cdot \nabla)\vec{f}_j(\vec{y})\Big|_{\vec{y}_n} \cdot \nabla)\vec{f}_i(\vec{y}) \Big\}_{\vec{y}_n} +$$

$$+ O(h_n^4).$$

A comparison of the expansions (2.2") and (2.3') yields the order conditions listed in table 2.1.

Table 2.1. Order conditions for scheme (2.1)

p = 1	$\sum_{\ell=0}^{m} \lambda_{m\ell i} = 1, \quad i = 1(1)k,$
p = 2	$\sum_{\ell=1}^{m} \sum_{r=0}^{\ell} \lambda_{m\ell i}\lambda_{\ell rj} = \frac{1}{2}, \quad i,j = 1(1)k,$
p = 3	$\sum_{r=0}^{\ell} \lambda_{\ell rj} = c_\ell, \quad j = 1(1)k,$
	$\sum_{\ell=1}^{m} \lambda_{m\ell i}\, c_\ell^2 = \frac{1}{3}, \quad i = 1(1)k,$
	$\sum_{\ell=1}^{m} \sum_{r=1}^{\ell} \sum_{s=0}^{r} \lambda_{m\ell i}\lambda_{\ell rj}\lambda_{rst} = \frac{1}{6}, \quad i,j,t = 1(1)k.$

REMARK. In this paper we do not give a convergence proof of method (2.1). It is observed that (2.1) is a one-step integration method of the type $\vec{y}_{n+1} = \vec{y}_n + h_n \vec{\Phi}(h_n, \vec{y}_n, \vec{y}_{n+1})$, $\vec{\Phi}$ denoting the increment function. Convergence results for one-step methods defined by general increment functions are well known in the literature (see e.g. HENRICI [9] or STETTER [25]).

2.2. The amplification matrix and the stability function

A widely accepted approach in the stability analysis of integration methods for systems of ordinary differential equations is to analyse the stability of the methods for linear systems

$$\frac{d\vec{y}}{dx} = J\vec{y}, \tag{2.4}$$

where J denotes a constant matrix of which the eigenvalues possess non-positive real parts. Following this approach we write (cf. (1.2))

$$J = \sum_{i=1}^{k} J_i, \tag{2.5}$$

and apply scheme (2.1) to equation (2.4). An elementary calculation yields

$$\vec{y}_{n+1} = R_m(h_n J_1, \ldots, h_n J_k)\vec{y}_n, \tag{2.6}$$

where the matrix R_m is defined by the formal relations

$$\begin{aligned} R_0(Z_1,\ldots,Z_k) &= I, \\ R_j(Z_1,\ldots,Z_k) &= I + \sum_{\ell=0}^{j} \sum_{i=1}^{k} \lambda_{j\ell i}\, Z_i\, R_\ell(Z_1,\ldots,Z_k), \\ & \qquad j = 1,\ldots,m. \end{aligned} \tag{2.7}$$

Here, I denotes the unit matrix and $Z_i = h_n J_i$, $i = 1,\ldots,k$. The matrix $R_m(Z_1,\ldots,Z_k)$ is called the *amplification matrix* of the scheme. In order to derive stability results for schemes from class (2.1) one has to investigate this matrix.

When performing such a stability investigation we must make assumptions about the matrices J_i. It is of course desirable that these assumptions are relevant for the differential equation under consideration. In this respect it is of importance to note that the matrices J_i stand for the Jacobian matrices of the functions $\vec{f}_i$. For example, when discussing semi-discretized partial differential equations a relevant assumption in the stability analysis is that the matrices J_i share the same eigensystem and are diagonizable. In conformity with the stability theory of integration methods for ordinary differential equations we then investigate the stability of the scalar recurrence relation

$$y_{n+1} = R_m(z_1,\ldots,z_k)y_n, \tag{2.8}$$

where z_i represents an eigenvalue of $h_n J_i$. The rational function $R_m: \mathbb{C}^k \to \mathbb{C}$ is accordingly called the *stability function* of the scheme. It shall be clear that the analysis of the stability function is simpler than the analysis of the amplification matrix, which is sometimes inpracticable.

2.3. An illustrative example of a two-term splitting method

In this section we derive a second order splitting method for equations which are splitted as (k = 2)

$$\vec{f}(\vec{y}) = \vec{f}_1(\vec{y}) + \vec{f}_2(\vec{y}). \tag{2.9}$$

We take m = 2, i.e. the method will use 2 stages. For p = 2 the order contions then read

$$\begin{aligned} &\lambda_{20i} + \lambda_{21i} + \lambda_{22i} = 1, \quad i = 1,2, \\ &\lambda_{21i}(\lambda_{10j} + \lambda_{11j}) + \lambda_{22i} = \tfrac{1}{2}, \quad i,j = 1,2. \end{aligned} \tag{2.10}$$

In order to exploit the splitting property (2.9) we choose $\lambda_{112} = \lambda_{221} = 0$. By this choice the implicitness of the scheme is distributed over two stages. For calculational convenience we further put $\lambda_{101} + \lambda_{111} = \lambda_{102} + \lambda_{112}$. This condition simplifies the second order conditions in (2.10). After some calculations we thus arrive at the following parameter matrices

$$(\lambda_{j\ell 1}) = \begin{pmatrix} \lambda_{101} & \lambda_{111} & 0 \\ \dfrac{2(\lambda_{101}+\lambda_{111})-1}{2(\lambda_{101}+\lambda_{111})} & \dfrac{1}{2(\lambda_{101}+\lambda_{111})} & 0 \end{pmatrix}, \tag{2.11}$$

$$(\lambda_{j\ell 2}) = \begin{pmatrix} \lambda_{101}+\lambda_{111} & 0 & 0 \\ \tfrac{1}{2} + \lambda_{212}(\lambda_{101}+\lambda_{111}-1) & \lambda_{212} & \tfrac{1}{2} - \lambda_{212}(\lambda_{101}+\lambda_{111}) \end{pmatrix},$$

where λ_{101}, λ_{111} and λ_{212} are still free parameters which can be used to adapt the stability of the scheme.

The formal expressions for the amplification matrix (cf. (2.7)) is given by (note that the matrices Z_1 and Z_2 are generally non-commutative)

$$R_2(Z_1,Z_2) = (I - \lambda_{222}\, Z_2)^{-1}\{(I + \lambda_{201}\, Z_1 + \lambda_{202}\, Z_2) +$$

$$(\lambda_{211}\, Z_1 + \lambda_{212}\, Z_2)(I - \lambda_{111}\, Z_1)^{-1} \tag{2.12}$$

$$(I + \lambda_{101}\, Z_1 + \lambda_{102}\, Z_2)\}.$$

Our choice $\lambda_{221} = \lambda_{112} = 0$ implies that the denominator of R_2 does not contain Z_1^2 and Z_2^2 terms. We prefer to be able to deal with functions $\vec{f}_i$ with large Lipschitz constants, such as functions originating from semi-discretization of a partial differential equation. Hence we will next require that in the numerator these terms also vanish. Substitution of the parameters listed in (2.11), and equating coefficients to zero, yields $\lambda_{111} = \frac{1}{2}$ and $\lambda_{212} = 0$, so that

$$R_2(Z_1,Z_2) = (I-\tfrac{1}{2}Z_2)^{-1}(I-\tfrac{1}{2}Z_1)^{-1}(I+\tfrac{1}{2}Z_1)(I+\tfrac{1}{2}Z_2). \tag{2.12'}$$

The matrices (2.11) become

$$(\lambda_{j\ell 1}) = \begin{pmatrix} \lambda-\frac{1}{2} & \frac{1}{2} & 0 \\ & & \\ \frac{2\lambda-1}{2\lambda} & \frac{1}{2\lambda} & 0 \end{pmatrix},$$

(2.11')

$$(\lambda_{j\ell 2}) = \begin{pmatrix} \lambda & 0 & 0 \\ & & \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

λ still being a free parameter. The two-term splitting scheme generated by (2.11') is given by

$$\vec{y}_{n+1}^{(1)} = \vec{y}_n + h_n[(\lambda-\tfrac{1}{2})\vec{f}_1(\vec{y}_n) + \lambda\vec{f}_2(\vec{y}_n) + \tfrac{1}{2}\vec{f}_1(\vec{y}_{n+1}^{(1)})].$$

(2.13)

$$\vec{y}_{n+1} = \vec{y}_n + h_n[(\frac{2\lambda-1}{2\lambda})\vec{f}_1(\vec{y}_n) + \tfrac{1}{2}\vec{f}_2(\vec{y}_n) + \frac{1}{2\lambda}\vec{f}_1(\vec{y}_{n+1}^{(1)}) + \tfrac{1}{2}\vec{f}_2(\vec{y}_{n+1})].$$

If we put $\lambda = \frac{1}{2}$, we finally obtain

$$\vec{y}_{n+1}^{(1)} = \vec{y}_n + \tfrac{1}{2}h_n\vec{f}_2(\vec{y}_n) + \tfrac{1}{2}h_n\vec{f}_1(\vec{y}_{n+1}^{(1)}),$$

(2.13')

$$\vec{y}_{n+1} = \vec{y}_{n+1}^{(1)} + \tfrac{1}{2}h_n\vec{f}_1(\vec{y}_{n+1}^{(1)}) + \tfrac{1}{2}h_n\vec{f}_2(\vec{y}_{n+1}).$$

REMARK 2.2. For the non-autonomous equation (1.3) the most natural extension of (2.13') is

$$\vec{y}_{n+1}^{(1)} = \vec{y}_n + \tfrac{1}{2}h_n\vec{f}_2(x_n,\vec{y}_n) + \tfrac{1}{2}h_n\vec{f}_1(x_n+\tfrac{1}{2}h_n,\vec{y}_{n+1}),$$

(2.13")

$$\vec{y}_{n+1} = \vec{y}_{n+1}^{(1)} + \tfrac{1}{2}h_n\vec{f}_1(x_n+\tfrac{1}{2}h_n,\vec{y}_{n+1}^{(1)}) + \tfrac{1}{2}h_n\vec{f}_2(x_{n+1},\vec{y}_{n+1}).$$

It should be noted however that the x-increments are not uniquely determined. For example, if we write $x_n + \frac{1}{2}h_n$ in all functions in (2.13") second order accuracy is preserved.

REMARK 2.3. The intermediate approximation $\vec{y}_{n+1}^{(1)}$ is a first order approximation to the local analytical solution at $x = x_n+\frac{1}{2}h_n$.

The particular schemes from class (2.1), which are discussed in this paper, may all be recognized as existing schemes, or as generalizations of existing schemes, when we confine ourselves to a particular problem class.

To illustrate this we give three applications of scheme (2.13'), or its equivalent (2.13"), which are known in the literature.

2.3.1. The odd-even hopscotch method

Let f^i and y^i denote the i-th component of the vector functions $\vec{f}$ and $\vec{y}$, respectively. Assume that $\vec{f}$ originates from semi-discretization of a one- or multi-dimensional parabolic partial differential equation which may be linear or non-linear. Assume further that each component f^i, i odd (even), only depends on y^i and y^j $j \neq i$, with j even(odd). Next define the functions $\vec{f}_1$ and $\vec{f}_2$ in (2.9) by

$$\vec{f}_1^{\,i}(\vec{y}) = \begin{cases} \vec{f}^{\,i}(\vec{y}), & \text{i odd,} \\ 0, & \text{i even,} \end{cases}$$

$$\vec{f}_2^{\,i}(\vec{y}) = \begin{cases} \vec{f}^{\,i}(\vec{y}), & \text{i even,} \\ 0, & \text{i odd.} \end{cases}$$

Then, method (2.13') becomes an odd-even hopscotch method for semi-discretized equations (see GOURLAY [6], see also VERWER [15]).

2.3.2. The alternating direction method of Peaceman and Rachford

Assume that $\vec{f}$ originates from semi-discretization of a two-dimensional parabolic equation of which the differential operator can be written as a sum of two one-dimensional operators. Let $\vec{f}_1$ and $\vec{f}_2$ represent these two semi-discretized operators. Then, method (2.13') becomes an alternating direction method of the Peaceman-Rachford type for semi-discretized equations (see PEACEMAN & RACHFORD [12]). Here it is assumed that in (2.13') the semi-discretization gives rise to systems of non-linear algebraic equations with a tridiagonal Jacobian matrix.

2.3.3. A method of Samarskii

Assume that in (2.9) $\vec{f}_1$ and $\vec{f}_2$ can be defined such that $\partial\vec{f}_1/\partial\vec{y}$ and $\partial\vec{f}_2/\partial\vec{y}$ are triangular and negative definite. Then, we obtain an algorithm

of a type suggested by SAMARSKII [13] (Samarskii considers the linear case $\vec{f}(x,\vec{y}) = A(x)\vec{y} + \vec{g}(x)$).

2.3.4. Some stability theorems

We conclude this section with giving some known stability results on the amplification matrix (2.12'), which are of direct relevance to the applications mentioned above. For a precise interpretation of these results we refer to the given literature (see also YANENKO [16] for a discussion of stability). Remember that in (2.12') $Z_i = h_n J_i$ depends on the stepsize h_n, which is not necessarily constant (cf. section 2.2).

THEOREM 2.1. *Let the matrices* J_1 *and* J_2 *have the following properties:*

a) J_1 *and* J_2 *commute,*

b) $J_1 + J_1^T$ *and* $J_2 + J_2^T$ *are non-positive definite, then* $\|R_2(Z_1,Z_2)\|_2 \leq 1$, *for every stepsize* h_n.

This theorem expresses the desirable property of unconditional stability for varying stepsizes. When the matrices J_1 and J_2 are not constant, but satisfy the assumptions at each integration step, the theorem remains valid. For non-commuting matrices we have

THEOREM 2.2. *Let* $h_n = h$, h *constant. Let the matrices* $J_1 + J_1^T$ *and* $J_2 + J_2^T$ *be non-positive definite. Then* $R_2^n(Z_1,Z_2)$ *is uniformly bounded in* n *for every stepsize* h.

This theorem expresses the desirable property of unconditional stability for constant stepsizes. Here the matrices J_1 and J_2 must remain constant.

Both theorems immediately follow from

LEMMA 2.1. *Let* M *denote a square real matrix with transposed matrix* M^T, *such that* $M + M^T$ *is non-positive definite. Let* $\rho > 0$, *then*

a) $\rho I - M$ *is non-singular,*

b) $\|(\rho I - M)^{-1}\|_2 \leq \rho^{-1}$

c) $\|(\rho I + M)(\rho I - M)^{-1}\|_2 \leq 1$.

This lemma was originally posed by KELLOG [10], who stated and proved the lemma for linear operators on Hilbert spaces. For our, more simpler case, the lemma can be easily proved using elementary matrix theory.

Finally a result concerning the stability function (cf. (2.8)) related to (2.12').

THEOREM 2.3. $|R_2(z_1,z_2)| \leq 1$ *if* Re $z_1 \leq 0$ *and* Re $z_2 \leq 0$.

The proof follows immediately from the well-known result: $|(2 + z)/(2 - z)| \leq 1$ if Re $z \leq 0$.

3. OTHER EXAMPLES OF EXISTING SPLITTING METHODS

In section 2.3 we derived a two-term splitting method which, for the proper definition of the functions $\vec{f}_1$ and $\vec{f}_2$, appeared to be an alternating direction method of the Peaceman-Rachford type. In the present section we give some further examples of splitting methods which are known in the literature, and which all belong to class (2.1). Here, $\vec{f}$ is supposed to originate from semi-discretization of a k-dimensional parabolic equation of which the differential operator can be written as a sum of k one-dimensional operators. The functions $\vec{f}_i$ then should represent these one-dimensional operators which may be linear and non-linear.

In the present section we do not discuss specific stability properties of the various schemes, as this falls outside the scope of the paper. We observe that when the amplification matrix is factorized, results are most easily obtained using Kellog's lemma. In case of a non-factorized matrix results can be obtained by means of the stability function.

3.1. A three-term splitting method of Gourlay and Mitchell

Let k = m = 3, and define

$$\vec{y}_{n+1}^{(1)} = \vec{y}_n + \tfrac{1}{2}h_n\vec{f}_1(\vec{y}_n) + \tfrac{1}{2}h_n\vec{f}_2(\vec{y}_{n+1}^{(1)}),$$

$$(3.1)\qquad \vec{y}^{(2)}_{n+1} = \vec{y}^{(1)}_{n+1} + \tfrac{1}{2}h_n\vec{f}_3(\vec{y}^{(1)}_{n+1}) + \tfrac{1}{2}h_n\vec{f}_3(\vec{y}^{(2)}_{n+1}),$$

$$\vec{y}_{n+1} = \vec{y}^{(2)}_{n+1} + \tfrac{1}{2}h_n\vec{f}_2(\vec{y}^{(2)}_{n+1}) + \tfrac{1}{2}h_n\vec{f}_1(\vec{y}_{n+1}).$$

This type of method has been suggested by GOURLAY & MITCHELL [7]. A nice property of this three-stage scheme is that it is of second order, factorized, and unconditionally stable for a relevant class of problems. This is due to the combination of the alternating direction and locally one-dimensional principles. The generating matrices for (3.1) are given by

$$(\lambda_{j\ell 1}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$(3.2)\qquad (\lambda_{j\ell 2}) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

$$(\lambda_{j\ell 3}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

From these parameter matrices the second order conditions of table 2.1 are easily verified.

3.2. Examples of k-term splitting methods

In the present subsection m is always equal to k.

3.2.1. Locally one-dimensional methods of Yanenko

$$\vec{y}^{(0)}_{n+1} = \vec{y}_n,$$

$$(3.3)\qquad \vec{y}^{(j)}_{n+1} = \vec{y}^{(j-1)}_{n+1} + h_n(1-\alpha)\vec{f}_j(\vec{y}^{(j-1)}_{n+1}) + h_n\alpha\vec{f}_j(\vec{y}^{(j)}_{n+1}), \quad j = 1(1)k,$$

$$\vec{y}_{n+1} = \vec{y}^{(k)}_{n+1}.$$

Method (3.3) is called the locally one-dimensional method (YANENKO [16, p.23]). The free parameter α usually equals $\frac{1}{2}$ or 1. The method is of first order, which immediately follows from the parameter definition: $\lambda_{ji-1i} = 1-\alpha$, $\lambda_{jii} = \alpha$, $i = 1(1)j$, $j = 1(1)k$, and zero otherwise.

3.2.2. The method of approximation corrections of Yanenko

$$\begin{aligned}
&\vec{y}^{(0)}_{n+1} = \vec{y}_n, \\
&\vec{y}^{(j)}_{n+1} = \vec{y}^{(j-1)}_{n+1} + \tfrac{1}{2}h_n \vec{f}_j(\vec{y}^{(j)}_{n+1}), \quad j = 1(1)k-1, \qquad (3.4)\\
&\vec{y}_{n+1} = \vec{y}_n + h_n \sum_{i=1}^{k} \vec{f}_i(\vec{y}^{(k-1)}_{n+1}).
\end{aligned}$$

This type of method also proceeds from YANENKO [16, p.128], who called it the method of approximation corrections. It is charaterized by the fact that stability is achieved in the preliminary stages, while accuracy is obtained at the last stage. The method is of first order. The parameters are given by $\lambda_{jii} = \frac{1}{2}$, $i = 1(1)j$, $j = 1(1)k-1$; $\lambda_{kk-1i} = 1$, $i = 1(1)k$, and zero otherwise.

3.2.3. The method of stabilizing corrections of Douglas and Gunn

$$\begin{aligned}
&\vec{y}^{(1)}_{n+1} = \vec{y}_n + h_n \sum_{i=2}^{k} \vec{f}_i(\vec{y}_n) + h_n \vec{f}_1(\vec{y}^{(1)}_{n+1}), \\
&\vec{y}^{(j)}_{n+1} = \vec{y}^{(j-1)}_{n+1} - h_n \vec{f}_j(\vec{y}_n) + h_n \vec{f}_j(\vec{y}^{(j)}_{n+1}), \quad j = 2(1)k, \qquad (3.5)\\
&\vec{y}_{n+1} = \vec{y}^{(k)}_{n+1}.
\end{aligned}$$

For $k \leq 3$ this type of splitting method was introduced by DOUGLAS & RACHFORD [3], and later, in its general form, formulated by DOUGLAS & GUNN [5] (see also YANENKO [16, p.126]). Method (3.5) is called the method of stabilizing corrections. At the first stage, an accurate approximation is evaluated, while all succeeding stages are corrections and serve to improve the stability. Method (3.5) is also of first order, and its

parameters are given by $\lambda_{j0i} = 1$, $i = j+1(1)k$, $j = 1(1)k$; $\lambda_{jii} = 1$, $i = 1(1)j$, $j = 1(1)k$, and zero otherwise.

3.2.4. Generalized Douglas methods

$$
\begin{aligned}
&\vec{y}^{(1)}_{n+1} = \vec{y}_n + \tfrac{1}{2}h_n\vec{f}_1(\vec{y}_n) + h_n\sum_{i=2}^{k}\vec{f}_i(\vec{y}_n) + \tfrac{1}{2}h_n\vec{f}_1(\vec{y}^{(1)}_{n+1}),\\
&\vec{y}^{(j)}_{n+1} = \vec{y}^{(j-1)}_{n+1} - \tfrac{1}{2}h_n\vec{f}_j(\vec{y}_n) + \tfrac{1}{2}h_n\vec{f}_j(\vec{y}^{(j)}_{n+1}), \quad j = 2(1)k, \qquad (3.6)\\
&\vec{y}_{n+1} = \vec{y}^{(k)}_{n+1}.
\end{aligned}
$$

Method (3.6) is a second order analogue of method (3.5). The three-term scheme of this class originates from DOUGLAS [4]. As far as we know, the general case has not been discussed in the literature. The parameters are given by $\lambda_{j0i} = \frac{1}{2}$, $i = 1(1)j$; $\lambda_{j0i} = 1$, $i = j+1(1)k$; $\lambda_{jii} = \frac{1}{2}$, $i = 1(1)j$; for $j = 1(1)k$, and zero otherwise.

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